Journal of Approximation Theory **115**, 167–185 (2002)
doi:10.1006/jath.2001.3646, available online at http://www.idealibrary.com on IDEAL®

# Bounds and Inequalities for $L_m$ Extremal Polynomials[1]

Ying Guang Shi[2]

*Department of Mathematics, Hunan Normal University, Changsha, Hunan, China 410081*
E-mail: syg@lsec.cc.ac.cn

*Communicated by Doron S. Lubinsky*

Received May 22, 2001; accepted September 12, 2001

In this paper bounds and inequalities for $L_m$ extremal polynomials as well as their applications are given. 

*Key Words:* $L_m$ extremal polynomials, bounds; inequalities; mean convergence.

## 1. INTRODUCTION

Let $\alpha$ be a nondecreasing function on $I := [-1, 1]$ with infinitely many points of increase such that all moments of $d\alpha$ are finite. We call $d\alpha$ a measure. Let

$$\|f\| := \max_{x \in I} |f(x)|, \qquad f \in C[-1, 1],$$

and

$$(1.1) \qquad \|f\|_{d\alpha, m} := \left\{ \int_I |f|^m \, d\alpha \right\}^{1/m}, \qquad 0 < m < \infty, \qquad f \in L^m_{d\alpha}[-1, 1].$$

Then we define the $L_m$ monic extremal polynomials

$$(1.2) \qquad P_n^*(d\alpha, m, x) = x^n + \cdots, \qquad n = 0, 1, \ldots,$$

satisfying

$$(1.3) \qquad \|P_n^*(d\alpha, m)\|_{d\alpha, m} = \inf_{P(x) = x^n + \cdots} \|P\|_{d\alpha, m}$$

[1] Project 19971089 supported by National Natural Sciences Foundation of China.

[2] Current address: Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, People's Republic of China.



0021-9045/02 $35.00

and the $L_m$ normalized extremal polynomials

$$
\begin{aligned}
P_n(d\alpha, m, x) &= P_n^*(d\alpha, m, x)/\|P_n^*(d\alpha, m)\|_{d\alpha, m} \\
&= \gamma_n(d\alpha, m)\, x^n + \cdots, \qquad n = 0, 1, \ldots .
\end{aligned} \tag{1.4}
$$

Given a triangular matrix $X$ of nodes

$$1 \geqslant x_{1n} > x_{2n} > \cdots > x_{nn} \geqslant -1, \qquad n = 1, 2, \ldots, \tag{1.5}$$

denote the Lagrange interpolating polynomial of $f \in C[-1, 1]$ by

$$L_n(X, f, x) := \sum_{k=1}^{n} f(x_{kn})\, \ell_{kn}(X, x), \qquad n = 1, 2, \ldots, \tag{1.6}$$

and the Lebesgue function type sum by

$$S_n(X, x) := \sum_{k=1}^{n} |(x - x_{kn})\, \ell_{kn}(X, x)|, \qquad n = 1, 2, \ldots, \tag{1.7}$$

respectively, where the fundamental polynomials

$$\ell_{kn}(X, x) = \frac{\omega_n(x)}{\omega_n'(x_{kn})(x - x_{kn})}, \qquad k = 1, 2, \ldots, n, \qquad n = 1, 2, \ldots,$$

with $\omega_n(x) = (x - x_{1n})(x - x_{2n}) \cdots (x - x_{nn})$, $n = 1, 2, \ldots$ .

If $X$ consists of the zeros of $P_n(d\alpha, m)$ then we write $L_n(d\alpha, m, f)$ instead of $L_n(X, f)$, etc.

As we know, the case $m = 2$ is the special case of orthogonal polynomials; it has a long history of study and a classical theory. In contrast to this special case the theory for the general case is still in the developing stages.

In attempting to study convergence of orthogonal Fourier series or convergence of Lagrange interpolation at zeros of orthogonal polynomials, one invariably encounters the need for bounds and inequalities on the orthogonal polynomials on the interval of orthogonality. Historically, the problem of finding bounds and inequalities has lived under the shadow of the deeper asymptotics on the segment, for the latter are often the only way of obtaining the former. Of course, this way usually gives asymptotic estimates for certain "nice" weights only. Recently the author in [4, 5] has developed an effective approach—so called the non-asymptotic appoach to find bounds and inequalities of many important quantities in orthogonal polynomials for arbitrary measures. Using this appoach the author [6] also gets some elementary results for $L_m$ extremal polynomials. On the other hand, in [7] the author introduces the Christoffel type functions for $L_m$ extremal polynomials for $m \in \mathbf{N}_2$, where $\mathbf{N}_2$ stands for the set of even natural numbers. It turns out that this is a useful tool and will have many applications. The main aim of this paper is to use both the non-asymptotic appoach and the Christoffel type functions for $L_m$ extremal polynomials to establish more bounds and inequalities for $L_m$ extremal polynomials in details and to give their applications.

## 2. PRELIMINARIES

We introduce the following definitions and notations:

$$\mathbf{P}_n := \text{the set of all polynomials of degree at most } n;$$

$$Z(\alpha') := \{x \in I : \alpha'(x) = 0\};$$

$$\mathscr{M} := \text{the collection of all Lebesgue measurable sets in } I;$$

$$|\Omega| := \text{the measure of } \Omega, \qquad \Omega \in \mathscr{M};$$

$$\sigma(d\alpha, \Delta, \delta) := \frac{\inf\limits_{\substack{\Omega \in \mathscr{M} \\ \Omega \subset \Delta \\ |\Omega| = \delta}} \int_\Omega d\alpha(x)}{\int_\Delta d\alpha(x)}, \qquad \Delta \in \mathscr{M}, \quad 0 < \delta \leqslant |\Delta|,$$

$$\sigma(d\alpha, \delta) := \sigma(d\alpha, I, \delta).$$

We need some auxiliary lemmas.

Lemma A [5, Lemma 1]. *Let $d\alpha$ be an arbitrary measure supported in $[-1, 1]$ and $\Delta \in \mathscr{M}$. If $\int_\Delta d\alpha(x) > 0$, then there exists a number $\delta := \delta(d\alpha, \Delta)$, $0 < \delta \leqslant |\Delta|$ (in case $|\Delta \cap Z(\alpha')| < |\Delta|$ every $\delta$ satisfying $|\Delta \cap Z(\alpha')| < \delta \leqslant |\Delta|$ is suitable), such that $\sigma(d\alpha, \Delta, \delta) > 0$.*

Lemma B [4, Theorem 1]. *For any matrix $X$ and for any sequence of positive numbers $\{\varepsilon_n\}$ there exist sets $I_n \subset I$ such that $|I_n| \leqslant \varepsilon_n$ and*

$$\sum_{k=1}^{n} S_n(X, x) \geqslant \frac{\varepsilon_n}{24} \tag{2.1}$$

*holds for all $x \in I \setminus I_n$ and $n = 1, 2, \ldots$ .*

In what follows we denote by $x_{kn} = x_{kn}(d\alpha, m)$, $k = 1, 2, \ldots, n$, the zeros of the $L_m$ extremal polynomial $P_n(d\alpha, m)$ and for convenience we accept the notations $P_n(d\alpha) := P_n(d\alpha, 2)$, $\gamma_n(d\alpha) := \gamma_n(d\alpha, 2)$, etc. The letters $c$, $c_1$, ...

stand for positive constants independent of variables and indices, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas.

DEFINITION 2.1 [7]. For $m \in \mathbf{N}_2$ the Christoffel type function of $L_m$ extremal polynomials $\lambda_{m-2,n,m}(d\alpha, x)$ is defined by

$$\lambda_{m-2,n,m}(d\alpha, x) = \min_{P \in \mathbf{P}_{n-1},\, P(x)=1} \frac{1}{(m-2)!} \int_{-1}^{1} P(t)^m (t-x)^{m-2}\, d\alpha(t). \tag{2.2}$$

*Remark.* In fact, in [7] the Christoffel type functions for $L_m$ extremal polynomials $\lambda_{j,n,m}(d\alpha, x)$ with $m \in \mathbf{N}_2$ are defined for $j = 0, 2, 4, \ldots, m-2$. But here we only need the special case when $j = m-2$ and use an alternative definition (see [7, Theorem 1]).

DEFINITION 2.2 [3, p. 106]. For $0 < m < \infty$ the generalized Christoffel function $\lambda_n(d\alpha, m, x)$ is defined by

$$\lambda_n(d\alpha, m, x) = \min_{P \in \mathbf{P}_{n-1},\, P(x)=1} \int_{-1}^{1} |P(t)|^m\, d\alpha(t). \tag{2.3}$$

Clearly, both the cases when $m = 2$ become the classical Christoffel function

$$\lambda_n(d\alpha, x) = \lambda_{0,n,2}(d\alpha, x) = \lambda_n(d\alpha, 2, x).$$

In this case we have

$$\lambda_n(d\alpha, x) = \left[ \sum_{k=1}^{n} \frac{\ell_{kn}(d\alpha, x)^2}{\lambda_{kn}(d\alpha)} \right]^{-1}, \tag{2.4}$$

where

$$\lambda_{kn}(d\alpha) = \lambda_n(d\alpha, x_{kn}), \qquad k = 1, 2, \ldots, n.$$

## 3. BOUNDS AND INEQUALITIES

We use the ideas of Lubinsky and Saff in the proof of [2, Lemma 3.1] which gives the relationship between $L_m$ extremal polynomials and classical orthogonal polynomials ($m = 2$). Let for $2 \leqslant m < \infty$

$$d\alpha_n(x) := |P_n(d\alpha, m, x)|^{m-2}\, d\alpha(x).$$

Then we have [6]

$$P_n(d\alpha_n, x) = P_n(d\alpha, m, x), \qquad \gamma_n(d\alpha_n) = \gamma_n(d\alpha, m),$$
$$x_{kn}(d\alpha_n) = x_{kn}(d\alpha, m), \qquad k = 1, 2, \ldots, n. \tag{3.1}$$

We need a basic expression of orthogonal polynomials with respect to the measure $d\alpha_n$ [3, p. 6]

$$\frac{\gamma_{n-1}(d\alpha_n)}{\gamma_n(d\alpha_n)} \lambda_{kn}(d\alpha_n)\, P_{n-1}(d\alpha_n, x_{kn})\, P_n(d\alpha_n, x)$$
$$= (x - x_{kn})\, \ell_{kn}(d\alpha_n, x), \qquad k = 1, 2, \ldots, n, \tag{3.2}$$

where $\lambda_{kn}(d\alpha_n) = \lambda_n(d\alpha_n, x_{kn})$. Hence

$$\frac{\gamma_{n-1}(d\alpha_n)}{\gamma_n(d\alpha_n)} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_n)\, |P_{n-1}(d\alpha_n, x_{kn})|\, |P_n(d\alpha_n, x)| = S_n(d\alpha_n, x). \tag{3.3}$$

The first task is further to investigate the relationship between the $L_m$ extremal polynomials with respect to $d\alpha$ and the classical orthogonal polynomials with respect to $d\alpha_n$.

LEMMA 3.1. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $2 \leqslant m < \infty$. Then*

$$\gamma_{n-1}(d\alpha, m) \leqslant \gamma_{n-1}(d\alpha_n) \leqslant \gamma_n(d\alpha_n) = \gamma_n(d\alpha, m). \tag{3.4}$$

*Proof.* To prove the first inequality of (3.4) we use the definition of orthogonal polynomials to obtain

$$\begin{aligned} S &= \int_{-1}^{1} P_{n-1}(d\alpha, m, x)^2 \, d\alpha_n(x) \\ &= \gamma_{n-1}(d\alpha, m)^2 \int_{-1}^{1} \left[ \frac{P_{n-1}(d\alpha, m, x)}{\gamma_{n-1}(d\alpha, m)} \right]^2 d\alpha_n(x) \\ &\geqslant \gamma_{n-1}(d\alpha, m)^2 \int_{-1}^{1} \left[ \frac{P_{n-1}(d\alpha_n, x)}{\gamma_{n-1}(d\alpha_n)} \right]^2 d\alpha_n(x) \\ &= \frac{\gamma_{n-1}(d\alpha, m)^2}{\gamma_{n-1}(d\alpha_n)^2}. \end{aligned}$$

On the other hand, by Hölder inequality and by the definition of $L_m$ extremal polynomials we have

$$S=\int_{-1}^{1} P_{n-1}(d\alpha, m, x)^2\,|P_n(d\alpha, m, x)|^{m-2}\,d\alpha(x)$$
$$\leqslant\left[\int_{-1}^{1} |P_{n-1}(d\alpha, m, x)|^m\,d\alpha(x)\right]^{2/m}\left[\int_{-1}^{1} |P_n(d\alpha, m, x)|^m\,d\alpha(x)\right]^{(m-2)/m}$$
$$=1.$$

Hence (3.4) follows. The second inequality immediately follows from a well known fact $\gamma_{n-1}(d\alpha_n)/\gamma_n(d\alpha_n)\leqslant 1$. The equality of (3.4) follows from (3.1). ∎

LEMMA 3.2. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $m\in\mathbf{N}_2$. Then for $1\leqslant k\leqslant n$*

$$\lambda_{kn}(d\alpha_n)=(m-2)!\,\lambda_{k,m-2,n,m}(d\alpha)\,P_n'(d\alpha, m, x_{kn})^{m-2} \tag{3.5}$$

*and*

$$P_{n-1}(d\alpha_n, x_{kn})=\frac{\gamma_n(d\alpha_n)}{(m-2)!\,\gamma_{n-1}(d\alpha_n)\,\lambda_{k,m-2,n,m}(d\alpha)\,P_n'(d\alpha, m, x_{kn})^{m-1}}, \tag{3.6}$$

*where $\lambda_{k,m-2,n,m}(d\alpha)=\lambda_{m-2,n,m}(d\alpha, x_{kn})$.*

*Proof.* By definition

$$\lambda_{kn}(d\alpha_n)=\lambda_n(d\alpha_n, x_{kn})=\int_{-1}^{1}\ell_{kn}(d\alpha_n, x)^2\,d\alpha_n(x) \tag{3.7}$$
$$=\int_{-1}^{1}\ell_{kn}(d\alpha_n, x)^2\,P_n(d\alpha, m, x)^{m-2}\,d\alpha(x).$$

On the other hand, by Theorems 1 and 2 in [7]

$$\lambda_{k,m-2,n,m}(d\alpha)=\lambda_{m-2,n,m}(d\alpha, x_{kn})$$
$$=\frac{1}{(m-2)!}\int_{-1}^{1}\ell_{kn}(d\alpha, m, x)^m\,(x-x_{kn})^{m-2}\,d\alpha(x)$$
$$=\frac{1}{(m-2)!\,P_n'(d\alpha, m, x_{kn})^{m-2}}\int_{-1}^{1}\ell_{kn}(d\alpha, m, x)^2$$
$$\times P_n(d\alpha, m, x)^{m-2}\,d\alpha(x),$$

which, together with (3.7), yields (3.5).

Comparing the leading coefficients in both the sides of (3.2) we get (see [4, (20)])

$$\frac{\gamma_{n-1}(d\alpha_n)}{\gamma_n(d\alpha_n)}\,\lambda_{kn}(d\alpha_n)\,P_{n-1}(d\alpha_n,x_{kn})=\frac{1}{P_n'(d\alpha,m,x_{kn})},\tag{3.8}$$

which, coupled with (3.5), yields (3.6). ∎

**Lemma 3.3.** *Let $d\alpha$ be a measure supported in $[-1,1]$ and $m\in\mathbf{N}_2$. Then*

$$\lambda_n(d\alpha_n,x)=(m-2)!\left[\sum_{k=1}^{n}\frac{\ell_{kn}(d\alpha,m,x)^2}{\lambda_{k,m-2,n,m}(d\alpha)\,P_n'(d\alpha,m,x_{kn})^{m-2}}\right]^{-1}\tag{3.9}$$

*and*

(3.10)

$$P_{n-1}(d\alpha_n,x)=\frac{\gamma_n(d\alpha_n)}{(m-2)!\,\gamma_{n-1}(d\alpha_n)}\sum_{k=1}^{n}\frac{P_n(d\alpha,m,x)}{\lambda_{k,m-2,n,m}(d\alpha)\,P_n'(d\alpha,m,x_{kn})^m\,(x-x_{kn})}.$$

*Proof.* The formula (3.9) follows from (2.4) and (3.5). The formula (3.10) follows from (3.6) and the identity

$$P_{n-1}(d\alpha_n,x)=\sum_{k=1}^{n}P_{n-1}(d\alpha_n,x_{kn})\,\ell_{kn}(d\alpha_n,x).\quad ∎$$

**Lemma 3.4.** *Let $d\alpha$ be a measure supported in $[-1,1]$. Then*

$$\lambda_n(d\alpha_n,x)\leqslant\lambda_n(d\alpha,m,x)^{2/m},\qquad 2\leqslant m<\infty,\tag{3.11}$$

*and*

$$\lambda_n(d\alpha_n,x)\geqslant P_n(d\alpha,m,x)^{m-2}\,\lambda_{mn/2}(d\alpha,x),\qquad m\in\mathbf{N}_2.\tag{3.12}$$

*Proof.* By Definition 2.2 and Hölder inequality

$$\begin{aligned}\lambda_n(d\alpha_n,x)&=\min_{P\in\mathbf{P}_{n-1},\,P(x)=1}\int_{-1}^{1}|P(t)|^2\,d\alpha_n(t)\\&=\min_{P\in\mathbf{P}_{n-1},\,P(x)=1}\int_{-1}^{1}|P(t)|^2\,|P_n(d\alpha,m,x)|^{m-2}\,d\alpha(t)\\&\leqslant\min_{P\in\mathbf{P}_{n-1},\,P(x)=1}\left[\int_{-1}^{1}|P(t)|^m\,\alpha(t)\right]^{2/m}\\&\quad\times\left[\int_{-1}^{1}|P_n(d\alpha,m,x)|^m\,d\alpha(t)\right]^{(m-2)/m}\end{aligned}$$

$$= \left[ \min_{P \in \mathbf{P}_{n-1},\, P(x)=1} \int_{-1}^{1} |P(t)|^m \, \alpha(t) \right]^{2/m}$$
$$= \lambda_n(d\alpha, m, x)^{2/m}.$$

This proves (3.11).

To prove (3.12) let $P \in \mathbf{P}_{n-1}$ with $P(x) = 1$ satisfy

$$\lambda_n(d\alpha_n, x) = \int_{-1}^{1} |P(s)|^2 \, |P_n(d\alpha, m, s)|^{m-2} \, d\alpha(s).$$

By the definition of $\lambda_{mn/2}(d\alpha, x)$ we have

$$P(t)^2 \, P_n(d\alpha, m, t)^{m-2} \leqslant \lambda_{mn/2}(d\alpha, t)^{-1} \int_{-1}^{1} |P(s)|^2 \, |P_n(d\alpha, m, s)|^{m-2} \, d\alpha(s)$$
$$= \lambda_{mn/2}(d\alpha, t)^{-1} \, \lambda_n(d\alpha_n, x),$$

because $PP_n(d\alpha, m)^{(m-2)/2} \in \mathbf{P}_{(mn/2)-1}$. Inserting $t = x$ yields (3.12). ∎

Now let us establish bounds and inequalities for $L_m$ extremal polynomials.

THEOREM 3.1. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $2 \leqslant m < \infty$. Then there exists a number $\delta := \delta(d\alpha)$, $0 < \delta < 2$ (in case $|Z(\alpha')| < 2$ every $\delta$ satisfying $|Z(\alpha')| < \delta < 2$ is suitable), such that*

$$\frac{\gamma_{n-1}(d\alpha_n)}{\gamma_n(d\alpha_n)} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_n) \, |P_{n-1}(d\alpha_n, x_{kn})|$$
$$\geqslant \frac{2-\delta}{24} \, \sigma(d\alpha, \delta) \left[ \int_{-1}^{1} d\alpha(x) \right]^{1/m} > 0, \qquad n = 1, 2, \ldots . \tag{3.13}$$

*Proof.* By Hölder inequality it follows from (3.3) that

$$\int_{-1}^{1} S_n(d\alpha, m, x) \, d\alpha(x)$$
$$= \frac{\gamma_{n-1}(d\alpha_n)}{\gamma_n(d\alpha_n)} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_n) \, |P_{n-1}(d\alpha_n, x_{kn})| \int_{-1}^{1} |P_n(d\alpha, m, x)| \, d\alpha(x)$$
$$\leqslant \frac{\gamma_{n-1}(d\alpha_n)}{\gamma_n(d\alpha_n)} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_n) \, |P_{n-1}(d\alpha_n, x_{kn})| \left[ \int_{-1}^{1} d\alpha(x) \right]^{(m-1)/m}.$$

Meanwhile, let $\delta$, $0<\delta<2$, be given in Lemma A so that $\sigma(d\alpha, \delta)>0$. Then applying Lemma B with $\varepsilon_n=2-\delta$ we get $I_n \subset I$ such that $|I_n| \leqslant 2-\delta$ and

$$\int_{-1}^{1} S_n(d\alpha, m, x)\, d\alpha(x) \geqslant \frac{2-\delta}{24} \int_{I \setminus I_n} d\alpha(x).$$

Since $|I \setminus I_n| \geqslant \delta$, by the definition of $\sigma(d\alpha, \delta)$ we have

$$\int_{I \setminus I_n} d\alpha(x) \geqslant \sigma(d\alpha, \delta) \int_{-1}^{1} d\alpha(x) > 0.$$

Thus (3.13) follows. ▮

This result improves Theorem 2.4 in [6] and has many applications.

COROLLARY 3.1. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $2 \leqslant m < \infty$. Then there exists a number $\delta := \delta(d\alpha)$, $0<\delta<2$ (in case $|Z(\alpha')|<2$ every $\delta$ satisfying $|Z(\alpha')|<\delta<2$ is suitable), such that for $n \geqslant 1$*

$$0 < \frac{2-\delta}{24} \sigma(d\alpha, \delta) \leqslant \frac{\gamma_{n-1}(d\alpha_n)}{\gamma_n(d\alpha_n)} \leqslant 1. \tag{3.14}$$

*Moreover, we have*

$$c_1(d\alpha, m)\, |P_n(d\alpha, m, x)| \leqslant S_n(d\alpha, m, x) \leqslant c_2(d\alpha, m)\, |P_n(d\alpha, m, x)| \tag{3.15}$$

*and*

$$c_1(d\alpha, m) \leqslant \sum_{k=1}^{n} \frac{1}{|P_n'(d\alpha, m, x_{kn})|} \leqslant c_2(d\alpha, m). \tag{3.16}$$

*Proof.* By Hölder inequality and the Gaussian quadrature formulas

$$\begin{aligned} &\sum_{k=1}^{n} \lambda_{kn}(d\alpha_n)\, |P_{n-1}(d\alpha_n, x_{kn})| \\ &\quad \leqslant \left[\sum_{k=1}^{n} \lambda_{kn}(d\alpha_n)\right]^{1/2} \left[\sum_{k=1}^{n} \lambda_{kn}(d\alpha_n)\, |P_{n-1}(d\alpha_n, x_{kn})|^2\right]^{1/2} \\ &\quad = \left[\int_{-1}^{1} d\alpha_n(x)\right]^{1/2} \end{aligned} \tag{3.17}$$

$$=\left[\int_{-1}^{1}|P_n(d\alpha, m, x)|^{m-2}\,d\alpha(x)\right]^{1/2}$$

$$\leqslant\left\{\left[\int_{-1}^{1}d\alpha(x)\right]^{2/m}\left[\int_{-1}^{1}|P_n(d\alpha, m, x)|^{m}\,d\alpha(x)\right]^{(m-2)/m}\right\}^{1/2}$$

$$=\left[\int_{-1}^{1}d\alpha(x)\right]^{1/m},$$

which, together with (3.13), yields

$$\frac{\gamma_{n-1}(d\alpha_n)}{\gamma_n(d\alpha_n)}\geqslant\frac{2-\delta}{24}\,\sigma(d\alpha, \delta)>0.$$

The last inequality of (3.14) follows from (3.4).

Inequality (3.15) follows from (3.3), (3.13), (3.17), and (3.14).

Comparing the leading coefficients of (3.15) gives (3.16). ▮

THEOREM 3.2. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $m\in\mathbf{N}_2$. Then*

$$c(d\alpha)\leqslant\frac{1}{(m-2)!}\sum_{k=1}^{n}\frac{1}{\lambda_{k,m-2,n,m}(d\alpha)\,P'_n(d\alpha, m, x_{kn})^m}\leqslant 1 \tag{3.18}$$

*and*

$$c(d\alpha)\leqslant\frac{1}{(m-2)!}\sum_{k=1}^{n}\frac{1}{\lambda_{k,m-2,n,m}(d\alpha)\,P'_n(d\alpha, m, x_{kn})^m\,(1-x_{kn}^2)}\leqslant 2. \tag{3.19}$$

*Proof.* Comparing the leading coefficients in both the sides of (3.10) and using (3.14) gives (3.18).

To prove (3.19) we use (3.5) and (3.6) to obtain

$$\sum_{k=1}^{n}\frac{\lambda_{kn}(d\alpha_n)\,P_{n-1}(d\alpha_n, x_{kn})^2}{1-x_{kn}^2}$$

$$=\frac{\gamma_n(d\alpha_n)^2}{(m-2)!\,\gamma_{n-1}(d\alpha_n)^2}\sum_{k=1}^{n}\frac{1}{\lambda_{k,m-2,n,m}(d\alpha)\,P'_n(d\alpha, m, x_{kn})^m\,(1-x_{kn}^2)},$$

that is,

$$\frac{1}{(m-2)!}\sum_{k=1}^{n}\frac{1}{\lambda_{k,m-2,n,m}(d\alpha)\,P'_n(d\alpha, m, x_{kn})^m\,(1-x_{kn}^2)}$$
$$=\frac{\gamma_{n-1}(d\alpha_n)^2}{\gamma_n(d\alpha_n)^2}\sum_{k=1}^{n}\frac{\lambda_{kn}(d\alpha_n)\,P_{n-1}(d\alpha_n, x_{kn})^2}{1-x_{kn}^2}. \tag{3.20}$$

According to an inequality given by Freud in [1, Formula (24)]

$$\frac{\gamma_{n-1}(d\alpha_n)^2}{\gamma_n(d\alpha_n)^2} \sum_{k=1}^{n} \frac{\lambda_{kn}(d\alpha_n)\, P_{n-1}(d\alpha_n, x_{kn})^2}{1-x_{kn}^2} \leqslant 2 \tag{3.21}$$

we obtain the right inequality of (3.19). By (3.18) we get

$$\frac{1}{(m-2)!} \sum_{k=1}^{n} \frac{1}{\lambda_{k,\, m-2,\, n,\, m}(d\alpha)\, P_n'(d\alpha, m, x_{kn})^m\, (1-x_{kn}^2)}$$
$$\geqslant \frac{1}{(m-2)!} \sum_{k=1}^{n} \frac{1}{\lambda_{k,\, m-2,\, n,\, m}(d\alpha)\, P_n'(d\alpha, m, x_{kn})^m} \geqslant c(d\alpha). \quad ∎$$

As an immediate consequence of Theorem 3.2 we state

Corollary 3.2. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $m \in \mathbf{N}_2$. Then*

$$c(d\alpha) \leqslant \frac{1}{(m-2)!} \sum_{k=1}^{n} \frac{\int_{-1}^{1} [(x-x_{kn})\, \ell_{kn}(d\alpha, m, x)]^m\, d\alpha(x)}{\lambda_{k,\, m-2,\, n,\, m}(d\alpha)(1-x_{kn}^2)} \leqslant 2. \tag{3.22}$$

*Proof.* Multiplying (3.19) by the number $\int_{-1}^{1} P_n(d\alpha, m, x)^m\, d\alpha(x) = 1$ we obtain

$$c(d\alpha) \leqslant \frac{1}{(m-2)!} \sum_{k=1}^{n} \frac{\int_{-1}^{1} P_n(d\alpha, m, x)^m\, d\alpha(x)}{\lambda_{k,\, m-2,\, n,\, m}(d\alpha)\, P_n'(d\alpha, m, x_{kn})^m\, (1-x_{kn}^2)} \leqslant 2,$$

which is equivalent to (3.22) if we notice that

$$\frac{P_n(d\alpha, m, x)}{P_n'(d\alpha, m, x_{kn})} = (x-x_{kn})\, \ell_{kn}(d\alpha, m, x). \quad ∎$$

Theorem 3.3. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $m \in \mathbf{N}_2$. Then*

$$\lambda_n(d\alpha_n, x)\, P_n(d\alpha, m, x)^2 \leqslant c(d\alpha, m) \tag{3.23}$$

*and*

$$\lambda_{mn/2}(d\alpha, x)\, P_n(d\alpha, m, x)^m \leqslant c_1(d\alpha, m). \tag{3.24}$$

*Proof.* By (3.15) and (2.4)

$$P_n(d\alpha, m, x)^2 \leqslant cS_n(d\alpha, m, x)^2 \leqslant c\left[\sum_{k=1}^{n} |\ell_{kn}(d\alpha, m, x)|\right]^2$$
$$\leqslant c\left[\sum_{k=1}^{n} \lambda_{kn}(d\alpha_n)\right]\left[\sum_{k=1}^{n} \frac{\ell_{kn}(d\alpha, m, x)^2}{\lambda_{kn}(d\alpha_n)}\right]$$
$$\leqslant c\lambda_n(d\alpha_n, x)^{-1},$$

which is equivalent to (3.23). Further, (3.24) follows from (3.23) and (3.12). ∎

THEOREM 3.4. *Let $d\alpha$ and $d\beta$ be measures supported in $[-1, 1]$, $2 \leqslant m < \infty$, and $0 < p < \infty$. Then for $\Delta \in \mathcal{M}$ and $\delta < |\Delta|$*

(3.25)
$$\int_\Delta |P_n(d\alpha, m, x)|^p \, d\beta(x) \geqslant \frac{(|\Delta| - \delta)^p \, \sigma(d\beta, \Delta, \delta) \int_\Delta d\beta(x)}{(24)^p \, [\int_{-1}^{1} d\alpha(x)]^{p/m}}, \qquad n = 0, 1, \ldots .$$

*Proof.* using (3.3), (3.14), and (3.17) we obtain

$$\int_\Delta |P_n(d\alpha, m, x)|^p \, d\beta(x) \geqslant \left[\int_{-1}^{1} d\alpha(x)\right]^{-p/m} \int_\Delta |S_n(d\alpha, m, x)|^p \, d\beta(x).$$

Applying Lemma B with $\varepsilon_n \equiv |\Delta| - \delta$ we can choose $I_n$ so that $|I_n| \leqslant \varepsilon_n = |\Delta| - \delta$ and

$$\int_\Delta |S_n(d\alpha, m, x)|^p \, d\beta(x) \geqslant \frac{(|\Delta| - \delta)^p}{(24)^p} \int_{\Delta \setminus I_n} d\beta(x)$$
$$\geqslant \frac{(|\Delta| - \delta)^p}{(24)^p} \sigma(d\beta, \Delta, \delta) \int_\Delta d\beta(x).$$

Hence (3.25) follows. ∎

THEOREM 3.5. *Let $d\alpha$ and $d\beta$ be measures supported in $[-1, 1]$ and $2 \leqslant m < \infty$. If $\Delta \in \mathcal{M}$ satisfies $|\Delta \setminus Z(\beta')| > 0$, i.e.,*

$$\int_\Delta \beta'(x) \, dx > 0,$$

*then for* $0 < p < \infty$

$$\liminf_{n\to\infty} \int_\Delta |P_n(d\alpha, m, x)|^p \, d\beta(x) > 0. \tag{3.26}$$

*Moreover, if* $\beta$ *is absolutely continuous on* $\Delta$ *then the converse is true.*

*In particular,*

$$\liminf_{n\to\infty} \int_\Delta |P_n(d\alpha, m, x)|^p \, dx > 0. \tag{3.27}$$

*if and only if* $|\Delta| > 0$.

*Proof.* The inequality (3.26) follows by Lemma A from (3.25). The other conclusions of the theorem are immediate consequences of the first one. ∎

As direct consequences of this result by the same arguments as that of [4, Corollary 9 and Theorem 8] we state the following corollaries, omitting the details.

COROLLARY 3.3. *Let* $d\alpha$ *be a measure supported in* $[-1, 1]$, $2 \leqslant m < \infty$, *and* $0 < p < \infty$. *Then for any* $\Delta \in \mathcal{M}$

$$\int_\Delta |P_n(d\alpha, m, x)|^p \, dx \geqslant \frac{|\Delta|^{p+1}}{2(48)^p \left[\int_{-1}^1 d\alpha(x)\right]^{p/m}}, \qquad n = 0, 1, \ldots. \tag{3.28}$$

COROLLARY 3.4. *Let* $d\alpha$ *be a measure supported in* $[-1, 1]$ *and* $2 \leqslant m < \infty$. *Then for any sequence of positive numbers* $\mathscr{E} = \{\varepsilon_n\}$ *there exist sets*

$$I_n := I_n(\mathscr{E}, d\alpha) = \bigcup_{k=1}^{n} (x_{kn} - h_{kn}, x_{kn} + h'_{kn}) \bigcap [-1, 1]$$

*with* $h_k, h'_k > 0$ *such that* $|I_n| \leqslant \varepsilon_n$ *and*

$$|P_n(d\alpha, m, x)| \geqslant \frac{\varepsilon_n}{24\left[\int_{-1}^1 d\alpha(x)\right]^{1/m}}, \tag{3.29}$$

*holds for all* $x \in [-1, 1] \setminus I_n$ *and* $n = 1, 2, \ldots$.

## 4. APPLICATIONS

As applications of the previous results we discuss $L_p$ convergence of the two operators

$$F_n(d\alpha, m, f, x) = \lambda_n(d\alpha_n, x) \sum_{k=1}^{n} f(x_{kn}) \frac{\ell_{kn}(d\alpha_n, x)^2}{\lambda_{kn}(d\alpha_n)} \tag{4.1}$$

and

$$G_n(d\alpha, m, f, x) = \lambda_n(d\alpha_n, x) \int_{-1}^{1} f(t)\, K_n(d\alpha_n, x, t)^2\, d\alpha_n(t), \tag{4.2}$$

where

(4.3)

$$K_n(d\alpha_n, x, t) = \frac{\gamma_{n-1}(d\alpha_n)}{\gamma_n(d\alpha_n)} \cdot \frac{P_{n-1}(d\alpha_n, t)\, P_n(d\alpha_n, x) - P_{n-1}(d\alpha_n, x)\, P_n(d\alpha_n, t)}{x-t}.$$

The special cases when $m = 2$ are introduced by Nevai in [3, pp. 58, 74]. By the same arguments as that of [3, Properties 6.1.1, p. 58, and Properties 6.2.1, p. 74] we state the following

LEMMA 4.1. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $2 \leqslant m < \infty$.*

(a) *If $f(x) \equiv 1$ then $F_n(d\alpha, m, f, x) \equiv 1$.*

(b) *If $f(x) \geqslant 0$ for $x \in [-1, 1]$ then $F_n(d\alpha, m, f, x) \geqslant 0$ for $x \in [-1, 1]$.*

(c) *$F_n(d\alpha, m, f, x_{kn}) = f(x_{kn})$ for $k = 1, 2, ..., n$.*

(d) *$F_n'(d\alpha, m, f, x_{kn}) = 0$ for $k = 1, 2, ..., n$.*

(e) *$F_n(d\alpha, m, f)$ is a rational function of degree $(2n-2, 2n-2)$, only the numerator depends on $f$.*

(f) *If $f(x) \equiv 1$ then $G_n(d\alpha, m, f, x) \equiv 1$.*

(g) *If $f(x) \geqslant 0$ for $x \in [-1, 1]$ then $G_n(d\alpha, m, f, x) \geqslant 0$ for $x \in [-1, 1]$.*

(h) *$G_n(d\alpha, m, f)$ is a rational function of degree $(2n-2, 2n-2)$, only the numerator depends on $f$.*

We accept the notations $\alpha(1+0) = \alpha(1)$ and $\alpha(-1-0) = \alpha(-1)$. As auxiliary lemmas we give several results which are of independent interests.

LEMMA 4.2. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $0 < m < \infty$. Then $\alpha(t)$ is continuous at $t = x$ if and only if*

$$\lim_{n \to \infty} \lambda_n(d\alpha, m, x) = 0. \tag{4.4}$$

*Proof.* $(\Rightarrow)$ We give the proof for $x \in (-1, 1)$ only, the proof for $x \in \{-1, 1)$ being similar.

Assume that $\varepsilon > 0$ is arbitrary. By continuity there exists a positive number $\delta \leqslant \min\{\varepsilon, 1+x, 1-x, 1/2\}$ such that $|\alpha(t)-\alpha(x)| \leqslant \varepsilon/2$ whenever $|t-x| \leqslant \delta$. Choose $f \in C[-1, 1]$ so that

$$f(t) = \begin{cases} 1, & t \in [x-\delta/2, x+\delta/2], \\ 0, & t \in [-1, x-\delta] \cup [x+\delta, 1], \\ \text{a linear function}, & t \in [x-\delta, x-\delta/2], \\ \text{a linear function}, & t \in [x+\delta/2, x+\delta]. \end{cases}$$

By Weierstrass theorem there is a polynomial $P \in \mathbf{P}_{N-1}$ for $N$ large enough so that $\|P-f\| \leqslant \delta$. Then by Definition 2.2 we have

$$\begin{aligned} \lambda_N(d\alpha, m, x) &\leqslant \frac{1}{|P(x)|^m} \int_{-1}^{1} |P(t)|^m \, d\alpha(t) \\ &\leqslant \frac{1}{|f(x)-\delta|^m} \int_{-1}^{1} [f(t)+\delta]^m \, d\alpha(t) \\ &= \frac{1}{(1-\delta)^m} \left\{ \int_{-1}^{x-\delta} \delta^m \, d\alpha(t) + \int_{x+\delta}^{1} \delta^m \, d\alpha(t) \right. \\ &\quad \left. + \int_{x-\delta}^{x+\delta} [f(t)+\delta]^m \, d\alpha(t) \right\} \\ &\leqslant \frac{1}{(1-\delta)^m} \left\{ \delta^m \int_{-1}^{1} d\alpha(t) + (1+\delta)^m \int_{x-\delta}^{x+\delta} d\alpha(t) \right\} \\ &= \frac{1}{(1-\delta)^m} \{\delta^m[\alpha(1)-\alpha(-1)] + (1+\delta)^m [\alpha(x+\delta)-\alpha(x-\delta)]\} \\ &\leqslant 2^m\{\varepsilon^m[\alpha(1)-\alpha(-1)] + (3/2)^m \varepsilon\}. \end{aligned}$$

Since $\lambda_n(d\alpha, m, x) \leqslant \lambda_N(d\alpha, m, x)$ for $n \geqslant N$, we obtain

$$\lim_{n\to\infty} \lambda_n(d\alpha, m, x) \leqslant 2^m\{\varepsilon^m[\alpha(1)-\alpha(-1)] + (3/2)^m \varepsilon\}.$$

Letting $\varepsilon \to 0$ we get (4.4).

$(\Leftarrow)$ It suffices to show

$$\lambda_n(d\alpha, m, x) \geqslant \alpha(x+0) - \alpha(x-0). \tag{4.5}$$

To this end let $P \in \mathbf{P}_{n-1}$ with $P(x) = 1$ satisfy

$$\lambda_n(d\alpha, m, x) = \int_{-1}^{1} |P(t)|^m \, d\alpha(t).$$

Then for $\varepsilon > 0$

$$\begin{aligned}\lambda_n(d\alpha, m, x) &\geqslant \int_{I \cap [x-\varepsilon, x+\varepsilon]} |P(t)|^m \, d\alpha(t) \\ &\geqslant \min_{t \in I \cap [x-\varepsilon, x+\varepsilon]} |P(t)|^m \int_{I \cap [x-\varepsilon, x+\varepsilon]} d\alpha(t) \\ &\geqslant \min_{t \in I \cap [x-\varepsilon, x+\varepsilon]} |P(t)|^m \, [\alpha(x+0) - \alpha(x-0)],\end{aligned}$$

which yields (4.5) by letting $\varepsilon \to 0$, because $P(x) = 1$. ∎

Lemma 4.3. *Let $\delta\alpha$ be a measure supported in $[-1, 1]$ and $0 < m < \infty$. Then the following statements are equivalent:*

(a) $\alpha \in C[-1, 1]$;

(b) $\lim_{n \to \infty} \lambda_n(d\alpha, m, x) = 0$, $\forall x \in [-1, 1]$;

(c) $\lim_{n \to \infty} \|\lambda_n(d\alpha, m)\| = 0$;

(d) $\lim_{n \to \infty} \int_{-1}^{1} \lambda_n(d\alpha, m, x) \, \alpha(x) = 0$.

*Proof.* (a) $\Leftrightarrow$ (b). Apply Lemma 4.2.

(b) $\Leftrightarrow$ (c). Apply Dini theorem, since by Definition 2.2 $\lambda_n(d\alpha, m, x)$ is monotonically decreasing with respect to $n$ for each fixed $x$.

(a) $\Leftrightarrow$ (d). Trivial.

(d) $\Leftrightarrow$ (a). By (4.5) we get that for $x \in [-1, 1]$

$$\int_{-1}^{1} \lambda_n(d\alpha, m, x) \, d\alpha(x) \geqslant [\alpha(x+0) - \alpha(x-0)]^2, \tag{4.6}$$

which by Statement (d) implies Statement (a). ∎

Lemma 4.4. *Let $d\alpha$ be a measure supported in $[-1, 1]$, $2 < m < \infty$, and $0 < p < \infty$. If $\alpha \in C[-1, 1]$, then*

$$\lim_{n \to \infty} \int_{-1}^{1} [\lambda_n(d\alpha_n, x) \, P_n(d\alpha_n, x)^2]^p \, d\alpha_n(x) = 0 \tag{4.7}$$

*and*

$$\lim_{n \to \infty} \int_{-1}^{1} [\lambda_n(d\alpha_n, x) \, P_{n-1}(d\alpha_n, x)^2]^p \, d\alpha_n(x) = 0. \tag{4.8}$$

*Proof.* We give the proof of (4.7) only, the proof of (4.8) being similar. If $p \geqslant 1$ then by (3.23) and (3.11) with $c = c(d\alpha, m)$ defined in (3.23)

$$\begin{aligned}
&\int_{-1}^{1} [\lambda_n(d\alpha_n, x)\, P_n(d\alpha_n, x)^2]^p\, d\alpha_n(x) \\
&\qquad = c^p \int_{-1}^{1} \left[\frac{\lambda_n(d\alpha_n, x)\, P_n(d\alpha_n, x)^2}{c}\right]^p d\alpha_n(x) \\
&\qquad \leqslant c^p \int_{-1}^{1} \frac{\lambda_n(d\alpha_n, x)\, P_n(d\alpha_n, x)^2}{c}\, d\alpha_n(x) \\
&\qquad \leqslant c^{p-1} \int_{-1}^{1} \lambda_n(d\alpha, m, x)^{2/m}\, P_n(d\alpha_n, x)^2\, d\alpha_n(x) \\
&\qquad \leqslant c^{p-1}\, \|\lambda_n(d\alpha, m)\|^{2/m},
\end{aligned}$$

which implies (4.7).

If $p < 1$ then by Hölder inequality

$$\begin{aligned}
&\int_{-1}^{1} [\lambda_n(d\alpha_n, x)\, P_n(d\alpha_n, x)^2]^p\, d\alpha_n(x) \\
&\qquad \leqslant \left[\int_{-1}^{1} \lambda_n(d\alpha_n, x)\, P_n(d\alpha_n, x)^2\, d\alpha_n(x)\right]^p \left[\int_{-1}^{1} d\alpha_n(x)\right]^{1-p} \\
&\qquad \leqslant c \left[\int_{-1}^{1} \lambda_n(d\alpha_n, x)\, P_n(d\alpha_n, x)^2\, d\alpha_n(x)\right]^p,
\end{aligned}$$

which by the previous conclusion again implies (4.7). ∎

Now we can give our main result in this section as follows.

THEOREM 4.1. *Let $d\alpha$ be a measure supported in $[-1, 1]$, $2 < m < \infty$, and $0 < p < \infty$. If $\alpha \in C[-1, 1]$ and $f \in C[-1, 1]$, then*

$$\lim_{n \to \infty} \|F_n(d\alpha, m, f) - f\|_{d\alpha_n, p} = 0 \tag{4.9}$$

*and*

$$\lim_{n \to \infty} \|G_n(d\alpha, m, f) - f\|_{d\alpha_n, p} = 0 \tag{4.10}$$

*Proof.* First we point out that since $F_n(d\alpha, m, f)$ and $G_n(d\alpha, m, f)$ are linear positive operators and $F_n(d\alpha, m, 1, x) \equiv G_n(d\alpha, m, 1, x) \equiv 1$, in order to prove (4.9) and (4.10), it suffices to show

$$\lim_{n \to \infty} \|F_n(d\alpha, m, \phi_x, x)\|_{d\alpha_n, p} = 0 \tag{4.11}$$

and

$$\lim_{n\to\infty} \|G_n(d\alpha, m, \phi_x, x)\|_{d\alpha_n, p} = 0, \tag{4.12}$$

respectively, where $\phi_x(t) = (x-t)^2$.

To prove (4.11) by means of (3.2) and the Gaussian quadrature formulas

$$\begin{aligned}
&\|F_n(d\alpha, m, \phi_x, x)\|^p_{d\alpha_n, p}\\
&= \int_{-1}^{1} \left[\lambda_n(d\alpha_n, x) \sum_{k=1}^{n} \frac{(x-x_{kn})^2\, \ell_{kn}(d\alpha_n, x)^2}{\lambda_{kn}(d\alpha_n)}\right]^p d\alpha_n(x)\\
&= \int_{-1}^{1} \left[\lambda_n(d\alpha_n, x)\, P_n(d\alpha_n, x)^2 \frac{\gamma_{n-1}(d\alpha_n)^2}{\gamma_n(d\alpha_n)^2} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_n)\, P_{n-1}(d\alpha_n, x_{kn})^2\right]^p d\alpha_n(x)\\
&= \int_{-1}^{1} \left[\lambda_n(d\alpha_n, x)\, P_n(d\alpha_n, x)^2 \frac{\gamma_{n-1}(d\alpha_n)^2}{\gamma_n(d\alpha_n)^2}\right]^p d\alpha_n(x),
\end{aligned}$$

which by (4.7) tends to 0 as $n \to \infty$.

To prove (4.12) using (4.3) and (3.14) and applying the inequality $(|A|+|B|)^p \leqslant 2^p(|A|^p+|B|^p)$, we obtain

$$\begin{aligned}
&\|G_n(d\alpha, m, \phi_x, x)\|^p_{d\alpha_n, p}\\
&\quad= \int_{-1}^{1} \left\{\lambda_n(d\alpha_n, x) \int_{-1}^{1} (x-t)^2\, K_n(d\alpha_n, x, t)^2\, d\alpha_n(t)\right\}^p d\alpha_n(x)\\
&\quad= \int_{-1}^{1} \left\{\lambda_n(d\alpha_n, x) \int_{-1}^{1} \frac{\gamma_{n-1}(d\alpha_n)^2}{\gamma_n(d\alpha_n)^2} [P_{n-1}(d\alpha_n, t)\, P_n(d\alpha_n, x)\right.\\
&\qquad \left. - P_{n-1}(d\alpha_n, x)\, P_n(d\alpha_n, t)]^2\, d\alpha_n(t)\right\}^p d\alpha_n(x)\\
&\quad= \int_{-1}^{1} \left\{\lambda_n(d\alpha_n, x) \frac{\gamma_{n-1}(d\alpha_n)^2}{\gamma_n(d\alpha_n)^2} [P_n(d\alpha_n, x)^2 + P_{n-1}(d\alpha_n, x)^2]\right\}^p d\alpha_n(x),\\
&\quad\leqslant 2^p \int_{-1}^{1} \{[\lambda_n(d\alpha_n, x)\, P_n(d\alpha_n, x)^2]^p + [\lambda_n(d\alpha_n, x)\, P_{n-1}(d\alpha_n, x)^2]^p\}\, d\alpha_n(x),
\end{aligned}$$

which by (4.7) and (4.8) again tends to 0 as $n \to \infty$. ∎

## REFERENCES


1. G. Freud, On Hermite–Fejér interpolation processes, *Studia Sci. Math. Hungar.* **7** (1972), 307–316.
2. D. S. Lubinsky and E. B. Saff, Strong asymptotics for $L_p$ extremal polynomials ($1 < p \leqslant \infty$) associated with weights on $[-1, 1]$, *in* "Lecture Notes in Math.," Vol. 1287, pp. 83–104, Springer-Verlag, New York/Berlin, 1987.
3. P. Nevai, "Orthogonal Polynomials," Mem. Amer. Math. Soc., Vol. 213, Amer. Math. Soc., Providence, 1979.
4. Y. G. Shi, Bounds and inequalities for general orthogonal polynomials on finite intervals, *J. Approx. Theory* **73** (1993), 303–333.
5. Y. G. Shi, Bounds and inequalities for arbitrary orthogonal polynomials on finite intervals, *J. Approx. Theory* **81** (1995), 217–230.
6. Y. G. Shi, On $L_p$ extremal polynomials, *Const. Approx.* **14** (1998), 187–194.
7. Y. G. Shi, On Christoffel type functions for $L_p$ extremal polynomials, *J. Approx. Theory* **103** (2000), 281–291.


*Printed in Belgium*